MEMRISTOR HAMILTONIAN CIRCUITS

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We prove analytically that 2-element memristive circuits consisting of a passive linear inductor in parallel with a passive memristor, or an active memristive device, can be described explicitly by a Hamiltonian equation, whose solutions can be periodic or damped, and can be represented analytically by the constants of the motion along the circuit Hamiltonian. Generalizations to 3-element and 2N-element memristive Hamiltonian circuits are also presented where complex bifurcation phenomena including chaos, abound.

Keywords: Memristor; memristive devices; Hamiltonian; Pfaff's equation; Nosé-Hoover oscillator; Hénon and Heiles equation.

1. Introduction

From both classical and quantum mechanics, it is well known that the equations of motion of a conservative (i.e. lossless) physical system can be derived from a scalar function, or operator, called the Hamiltonian, representing the total energy of the system. In particular, each contour of the Hamiltonian of a mechanical system defines a periodic orbit of the system whose Hamiltonian is a constant of motion [Andronov et al., 1987]. In electrical circuits, Hamiltonian equations are formulated for lossless, nonlinear circuits [Chua & McPherson, 1974; Andronov et al., 1987; Bernstein & Liberman, 1989]. As a consequence, it is generally believed that dissipative systems do not have Hamiltonian equations because the trajectories are damped and hence not periodic.

We will show in this paper that memristors [Chua, 1971] and memristive devices [Chua & Kang, 1976] when connected with capacitors, and/or inductors, can be described by an analogous Hamiltonian whose contours are constants of motion. This result is counter-intuitive and quite surprising because memristors and memristive devices are basically dissipative devices, albeit endowed with memory. In this paper, we will present simple analytical examples of 2-element memristive circuits made of a passive linear inductor and a memristor, or a memristive device, and derive their Hamiltonian, and orbits, in explicit analytic form. We will show that 2-element memristive circuits have a continuum of periodic orbits, when the memristive device is active, and is hence capable of generating energy.\(^1\) Such memristive devices can be built.

\(^1\)Note that this memristive circuit consists of two elements. If the circuit includes more than two elements, there are many nonmemristive circuits which have a continuum of periodic orbits and is capable of generating energy.
with off-the-shelf components, and a power supply, as described in [Chua, 1971]. We will also exhibit a 2-element memristor circuit where the memristor is passive, and hence all trajectories are damped and must tend to the origin. What is truly astonishing here is that this 2-element memristor circuit has a Hamiltonian whose contours are precisely the damped trajectories! In other words, the Hamiltonian contours are not closed.

We will also present several 3-element memristive circuits by adding either a passive linear capacitor, or a sinusoidal voltage source, in series with the above cited 2-element memristive circuit. Once again, an explicit Hamiltonian will be derived for the 3-element inductor-capacitor-memristive-device circuit. Furthermore, we present a 2N-element memristive circuit, which can be recast into an N-degree of freedom Hamiltonian equation. We will also present numerical simulations of bifurcation diagrams which depict the evolution of rather complex nonlinear dynamics, including chaos.

2. 2-Element Memristive Circuits

Consider the circuit in Fig. 1, which consists of an inductor and a memristive device. The current-controlled memristive device is described by

\[
\begin{align*}
\frac{dx}{dt} &= f(i) g(x), \\
v &= M(x) i.
\end{align*}
\]

where

\[
H(x, i) = \int \frac{M(x)}{L g(x)} dx + \int \frac{f(i)}{i} di,
\]

and the symbol \( \int \) denotes the operation of primitive integral or indefinite integral. The Pfaff's equation (4) has the solution \( H(x, i) = H_0 \) (\( H_0 \) is any constant), since

\[
\frac{dH(x, i)}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial i} \frac{di}{dt} = 0.
\]

After time scaling by \( dt = (g(x)i) dt \) (for more details, see [Andronov et al., 1987; Nemytskii & Stepanov, 1989] and Appendix A), Eq. (4) assumes the equivalent form

\[
\begin{align*}
\frac{dx}{dt} &= f(i) g(x), \\
\frac{di}{dt} &= -M(x) i.
\end{align*}
\]
Note that $H_x$ is a function of $x$ alone, while $H_i$ is a function of $i$ alone. The Hamiltonian $H(x, i)$ is the sum of the pseudo-potential energy $H_x$ and the pseudo-kinetic energy $H_i$, that is,

$$H(x, i) = H_x(x) + H_i(i).$$  \hfill (12)

Example 1. Choose

$$f(i) = \alpha(i^2 - 1),$$
$$g(x) = 1,$$
$$M(x) = x + \beta,$$
$$L = 1,$$

where $\alpha$ and $\beta$ denote some constants. Then Eq. (2) is recast explicitly as

$$\frac{dx}{dt} = \alpha(i^2 - 1),$$
$$\frac{di}{dt} = -(x + \beta)i.$$  \hfill (13)

If $g(x)i = i \neq 0$, Eq. (14) can be transformed into the form of Pfaff's equation

$$\frac{1}{i} \left( \frac{dx}{dt} \right) = \alpha \left( i - \frac{1}{i} \right),$$
$$\frac{1}{i} \left( \frac{di}{dt} \right) = -(x + \beta).$$  \hfill (15)

After time scaling by $d\tau = (g(x)i) \, dt = idt$, we obtain the associated Hamiltonian system with respect to the scaled time $\tau$:

$$\frac{dx}{d\tau} = \alpha \left( i - \frac{1}{i} \right),$$
$$\frac{di}{d\tau} = -(x + \beta).$$  \hfill (16)

---

2 This time scaling maps orbits between systems (2) and (7) in a one-to-one manner except at the singularity $(x, i)$ where $g(x)i = 0$, although it may not preserve the time orientation of orbits.

3 In physical systems, kinetic and potential energy are traditionally denoted by $T$ and $V$, respectively. The Hamiltonian $H$ represents the total energy of the system, which is the sum of $T$ and $V$, that is, $H = T + V$.

4 In this example, the memristive device is active [Chua, 1971; Chua & Kang, 1976] because $M(x) < 0$ for $x < -\beta$.
The Hamiltonian of Eq. (16) is given by
\[
H(x, i) = \int \frac{M(x)}{Lg(x)} \, dx + \int \frac{f(i)}{i} \, di = \int (x + \beta) dx + \int \alpha \left( i - \frac{1}{i} \right) di = \begin{cases} 
\frac{x^2}{2} + \beta x + \alpha \left( \frac{i^2}{2} - \ln |i| \right), & i > 0 \\
\frac{x^2}{2} + \beta x + \alpha \left( \frac{i^2}{2} - \ln (-i) \right), & i < 0 
\end{cases} 
\]
\[= \frac{x^2}{2} + \beta x + \alpha \left( \frac{i^2}{2} - \ln |i| \right). \tag{17} \]

where \(i \neq 0\). Thus, Eq. (14) has the integral
\[
x^2 \frac{2}{2} + \beta x + \alpha \left( \frac{i^2}{2} - \ln |i| \right) = H_0, \tag{18} \]

where \(H_0\) is a constant and \(i \neq 0\).

The pseudo-potential energy and pseudo-kinetic energy are given respectively by
\[
H_{x}(x) = \int \frac{M(x)}{Lg(x)} \, dx = \frac{x^2}{2} + \beta x, \\
H_{i}(i) = \int \frac{f(i)}{i} \, di = \alpha \left( \frac{i^2}{2} - \ln |i| \right). \tag{19} 
\]

Recall that \(H(x, i) = H_{x}(x) + H_{i}(i)\), and \(dH(x, i)/(dt) = 0\), as illustrated below.

Fig. 2. Hamiltonian (17) with \(\alpha = 1, \beta = 0\), namely, \(H(x, i) = (1/2)x^2 + (1/2)i^2 - \ln |i|\), where \(i \neq 0\).

Fig. 3. Contours of Hamiltonian (17) of Hamiltonian circuit in Fig. 1 with \(\alpha = 1, \beta = 0\). Several closed curves along which the Hamiltonian has a constant value are illustrated on the \((x, i)\)-plane. Top and bottom figures correspond to contours on \(i > 0\) and on \(i < 0\), respectively. The \(x\)-axis is the horizontal axis and the \(i\)-axis is the vertical axis. Labels on contour lines denote the elevation (height) representing the total energy.

The Hamiltonian (17) and its contour plot for \(\alpha = 1, \beta = 0\) are shown in Figs. 2 and 3, respectively. Two trajectories of Eq. (14) with the same parameters are shown in Fig. 4, which move on
the contour

\[ H_0 = \frac{x(0)^2}{2} + \left( \frac{i(0)^2}{2} - \ln |i(0)| \right) \]

of the Hamiltonian, where the initial conditions are given by \( x(0) = 0.1, i(0) = \pm 0.1 \). The state \( x(t) \), current \( i(t) \), voltage \( v(t) \), power \( p(t) = v(t)i(t) \), and energy \( E(t) = \int_0^t p(t')dt' \) of the memristive device are shown in Fig. 5 as a function of the original unscaled time \( t \). We also show \( x(t), i(t), H_x(t), H_i(t), H(t) \) and \( E(t) \) as a function of time \( t \) in Fig. 6. Observe that the Hamiltonian \( H \) is conserved. A quick glimpse of Figs. 5(d) and 5(e) seems to suggest the memristor in this example generates energy for all times \( t \geq 0 \) in the sense that \( E(t) < 0 \). Our careful numerical analysis, shown in Fig. 7, however, reveals that \( E(t) > 0 \) over a very small neighborhood centered around \( t = 0 \), \( t = 5.9 \), \( t = 11.7 \) and \( t = 17.5 \). During these extremely brief time intervals where \( p(t) > 0 \), the memristor is absorbing the energy initially stored in the inductor where \( i(0) > 0 \). Hence, during each cycle, the memristor acts briefly like a sink and does not generate energy.

Fig. 4. Trajectories of Eq. (14) with \( \alpha = 1, \beta = 0 \). Initial conditions: \( x(0) = 0.1, i(0) = 0.1 \) (top) and \( x(0) = 0.1, i(0) = -0.1 \) (bottom). They are located in the neighborhood of the origin. Here, \( H_0 \approx 2.31 \).

Fig. 5. Numerical solutions of \( x(t), i(t), v(t), p(t) = v(t)i(t) \) and \( E(t) = \int_0^t p(t')dt' \) of Eq. (14) as a function of time \( t \) with initial conditions \( x(0) = 0.1, i(0) = 0.1 \), and parameters \( \alpha = 1, \beta = 0 \). Here, \( x(t) \) and \( i(t) \) are calculated numerically from Eq. (14) using the Mathematica function NDSolve (general numerical differential equation solver). Since Hamiltonian equations are known to be very sensitive to numerical round off errors, the numerical integration is shown only for a short time interval.
Remark. Since \( i(t) > 0 \) for all \( t > 0 \) in Fig. 5(b), the loci of \( i(t) \) versus \( v(t) \) will be a symmetrical nonintersecting “hysteresis loop”, and not a “pinched hysteresis loop” [Chua, 2011]. Observe also that since the memristance \( M(0) = 0 \) in Example 1, the memristor in this example is current controlled, but not voltage controlled because its memductance \( W(x) \) is infinite, and hence undefined, at \( x = 0 \).

Example 2. Choose
\[
\begin{align*}
    f(i) &= i + \beta, \\
    g(x) &= -x, \\
    M(x) &= \alpha(x^2 - 1), \\
    L &= 1,
\end{align*}
\]
where \( \alpha \) and \( \beta \) denote some constants.\(^5\) Then Eq. (2) can be recast as

\[\text{two-dimensional memristive system B} \quad \begin{align*}
\frac{dx}{dt} &= -(i + \beta)x, \\
\frac{di}{dt} &= -\alpha(x^2 - 1)i.
\end{align*} \tag{21}\]

If \(-g(x)i = -(ix) = ix \neq 0\), Eq. (21) can be transformed into the form of Pfaff’s equation

\[\text{Pfaff’s equation B} \quad \begin{align*}
\frac{1}{ix} \left( \frac{dx}{dt} \right) &= -\left( 1 + \frac{\beta}{i} \right), \\
\frac{1}{ix} \left( \frac{di}{dt} \right) &= -\alpha \left( x - \frac{1}{x} \right).
\end{align*} \tag{22}\]

\(^5\) In this example, the memristive device is also active since \( M(x) < 0 \) for \(|x| < 1\).
Fig. 7. Numerical solution of $E(t)$ over a brief interval in the neighborhood of $t \approx 0$, $t \approx 5.9$, $t \approx 11.7$ and $t \approx 17.5$. Observe that $E(t) > 0$ in these neighborhoods.
After time scaling by $d\tau = -(g(x)i)\,dt = ix\,dt$, we obtain the Hamiltonian system:

Hamiltonian system B

$$\begin{align*}
\frac{dx}{d\tau} &= -\left(1 + \frac{\beta}{x}\right), \\
\frac{di}{d\tau} &= -\alpha \left(x - \frac{1}{x}\right).
\end{align*}$$

(23)

In this case, the Hamiltonian of Eq. (23) is given by

$$H(x,i) = -\int \frac{M(x)}{Lg(x)}\,dx - \int \frac{f(i)}{i}\,di$$

$$= \int \alpha \left(x - \frac{1}{x}\right)\,dx - \int \left(1 + \frac{\beta}{x}\right)\,di$$

$$= \alpha \left(\frac{x^2}{2} - \ln|x|\right) - i - \beta\ln|i|. \quad (24)$$

where $ix \neq 0$. Thus, Eq. (21) has the integral

$$\alpha \left(\frac{x^2}{2} - \ln|x|\right) - i - \beta\ln|i| = H_0, \quad (25)$$

where $H_0$ is a constant and $ix \neq 0$. The pseudo-potential energy and the pseudo-kinetic energy are given respectively by

$$\begin{align*}
H_x(x) &= \alpha \left(\frac{x^2}{2} - \ln|x|\right), \\
H_i(i) &= -i - \beta\ln|i|.
\end{align*}$$

(26)

The Hamiltonian (24) and its contour plot for $\alpha = 1, \beta = 0.6$ are shown in Figs. 8 and 9, respectively. Two trajectories of Eq. (21) with the same parameters are shown in Fig. 10, which move on the contour

$$H_0 = \frac{x(0)^2}{2} - \ln|x(0)| - i(0) - 0.6\ln|i(0)|$$

$$= \frac{0.8^2}{2} - \ln0.8 - 4.0 - 0.6\ln4.0 \approx 3.71,$$

of the Hamiltonian, where the initial conditions are given by $x(0) = \pm0.8$, $i(0) = -4.0$. The state $x(t)$, current $i(t)$, voltage $v(t)$, power $p(t) = v(t)i(t)$, and energy $E(t) = \int_0^t p(t')\,dt'$ of the memristive device

are shown as a function of time $t$ in Fig. 11, in which $E(t)$ is positive except in a small neighborhood of $t \approx 10$ and $t \approx 20$. Thus, the memristive device behaves like a passive nonlinear capacitor most of the time. However, it becomes active, that is, $E(t) \approx -0.2 < 0$, in the neighborhood of $t \approx 10$ and $t \approx 20$ (see Fig. 12), where the memristive device generates energy. We also show $x(t), i(t), H_x(t), H_i(t), H(t)$ and $E(t)$ as a function of time $t$ in Fig. 13. Observe that the Hamiltonian $H$ is conserved.

Example 3. Choose

$$\begin{align*}
f(i) &= i, \\
g(x) &= 1, \\
M(x) &= x^2 + 1, \\
L &= 1.
\end{align*}$$

(27)

If we set $x = q$, the memristance $M(q) = q^2 + 1$ defines a passive memristor with $\varphi = q + (q^3/3)$ because $M(x) > 0$ for all $x$ and $i$. Here, $q$ and $\varphi$ denote a charge and a flux, respectively. In this example, Eq. (2) assumes the form

$$\begin{align*}
\frac{dx}{dt} &= i, \\
\frac{di}{dt} &= -(x^2 + 1)i.
\end{align*}$$

(28)
Fig. 9. Contours of the Hamiltonian (24) of the Hamiltonian circuit in Fig. 1 with \( \alpha = 1, \beta = 0.6 \). Several curves along which the Hamiltonian has a constant value are illustrated on the \((x,i)\)-plane. The \(x\)-axis is the horizontal axis and the \(i\)-axis is the vertical axis. Labels on contour lines denote the elevation (height) representing the total energy: (a) \( x < 0, i > 0 \), (b) \( x > 0, i > 0 \), (c) \( x < 0, i < 0 \) and (d) \( x > 0, i < 0 \).

If \( g(x)i = i \neq 0 \), Eq. (28) can be transformed into the form of Pfaff’s equation

\[
\text{Pfaff’s equation C}
\begin{align*}
\frac{1}{i} \left( \frac{dx}{dt} \right) &= 1, \\
\frac{1}{i} \left( \frac{di}{dt} \right) &= -(x^2 + 1).
\end{align*}
\] (29)

After time scaling by \( d\tau = (g(x)i) \ dt = idt \), we obtain the Hamiltonian system:

\[
\text{Hamiltonian system C}
\begin{align*}
\frac{dx}{d\tau} &= 1, \\
\frac{di}{d\tau} &= -(x^2 + 1).
\end{align*}
\] (30)

Fig. 10. Trajectories of Eq. (21) with \( \alpha = 1, \beta = 0.6 \). Initial conditions: \( x(0) = -0.8, i(0) = -4.0 \) (left) and \( x(0) = 0.8, i(0) = -4.0 \) (right). Here, \( H_0 \approx 3.71 \). Two black arrowheads indicate the initial points (conditions) of trajectories on the contour \( H_0 \approx 3.71 \).
Fig. 11. Numerical solutions of $x(t), i(t), v(t), p(t) = v(t)i(t)$ and $E(t) = \int_0^t p(t')dt'$ of Eq. (21) as a function of time $t$, with initial conditions $x(0) = 0.8, i(0) = -4.0$ and parameters $\alpha = 1, \beta = 0.6$.

Fig. 12. Numerical solution of $E(t)$ for the period $8 \leq t \leq 22$. Observe that $E(t) < 0$ in the neighborhood of $t \approx 10$ and $t \approx 20$.

The Hamiltonian of Eq. (30) is given by

$$H(x, i) = \int (x^2 + 1)dx + \int 1di$$

$$= \frac{x^3}{3} + x + i,$$

where $i \neq 0$. Thus, Eq. (28) has the integral

$$\frac{x^3}{3} + x + i = H_0,$$

where $H_0$ is a constant and $i \neq 0$. The pseudo-potential energy and the pseudo-kinetic energy are given respectively by

$$H_x(x) = \frac{x^3}{3} + x,$$

$$H_i(i) = i.$$

From Eq. (28), we obtain

$$\frac{d(i^2)}{dt} = 2i\frac{di}{dt} = -2(x^2 + 1)i^2 < 0 \quad (i \neq 0).$$

Hence, $i(t) \rightarrow 0$ as $t \rightarrow \infty$. Since the $x$-axis is a continuum of equilibrium points defined by $i = 0$, $i(t)$ satisfies

$$i(t) > 0, \quad \text{if } i(0) > 0,$$
and

\[ i(t) < 0, \quad \text{if} \ i(0) < 0, \quad (36) \]

for \( 0 < t < \infty \).

The Hamiltonian (31) and its contour plot are shown in Figs. 14 and 15, respectively. The vector field and trajectories associated with Eq. (28) are shown in Figs. 16 and 17, respectively. Two trajectories in Fig. 17 move on the contour

\[
H_0 = \frac{x(0)^3}{3} + x(0) + i(0) \\
= \frac{0.8^3}{3} + 0.8 - 4.0 \approx 3.03,
\]

of the Hamiltonian, where the initial conditions are given by \( x(0) = \pm 0.8, \ i(0) = -4.0 \). Observe that every trajectory tends to the \( x \)-axis, which is a continuum of equilibrium points. The state \( x(t) \), current \( i(t) \), voltage \( v(t) \), power \( p(t) \), and energy \( E(t) \) of the memristor are shown in Fig. 18 as a function of time \( t \). Observe that energy is dissipated in the memristor, since \( p(t) \) and \( E(t) \) are given respectively by

\[
p(t) = v(t)i(t) = M(x(t))i(t)^2 \\
= (x(t)^2 + 1)i(t)^2 \geq 0,
\]

and

\[
E(t) = \int_0^t p(t')dt' \\
= \int (x(t')^2 + 1)i(t')^2 dt' \geq 0. \quad (38)
\]

Figure 19 shows \( x(t), i(t), H_x(t), H_i(t), H(t) \) and \( E(t) \) as a function of time \( t \). Observe that the Hamiltonian is conserved, since \( H_x(x(t)) \neq 0 \) even though \( i(t) = 0 \) and \( v(t) = 0 \) as \( t \to \infty \).
Fig. 14. Hamiltonian (31). Red line denotes the trajectory with the elevation $H(x, i) = 0$.

Fig. 15. Contours of the Hamiltonian (31). Several curves along which the Hamiltonian has a constant value are illustrated on the $(x, i)$-plane. Red curve denotes the trajectory with $H(x, i) = 0$. The horizontal axis is the $x$-axis and the vertical axis is the $i$-axis. Labels on contour lines denote the elevation (height).

Fig. 16. Vector field defined by Eq. (28).

Fig. 17. Two typical trajectories of Eq. (28) which tend to the $x$-axis moving on the contour $H_0 \approx 3.03$. Initial conditions $x(0) = -0.8, i(0) = 4$ (top) and $x(0) = 0.8, i(0) = -4$ (bottom).
We can generalize Example 3 and prove the following:

2-element Memristor Circuit Hamiltonian Property 2
If the memristor in the parallel memristor-inductor circuit in Fig. 20 is an ideal charge-controlled passive memristor described by \( \varphi = \dot{\varphi}(q) \), then the Hamiltonian of the circuit is given explicitly by

\[
H(q, i) = \frac{\dot{\varphi}(q)}{L} + i. \tag{39}
\]

There is a continuum of equilibrium points defined by \( i = 0 \), independent of the initial charge \( q = q_0 \). Hence,

\[
i(t) > 0, \quad \text{if } i(0) > 0; \tag{40}
\]

and

\[
i(t) < 0, \quad \text{if } i(0) < 0; \tag{41}
\]

for \( 0 < t < \infty \). Moreover, an open-circuited memristor retains its initial charge for all times, and is therefore a nonvolatile memory.

2-element Memristor Circuit Hamiltonian Property 3
If the memristor in the parallel memristor-capacitor circuit in Fig. 21 is an ideal flux-controlled passive memristor described by \( q = \dot{q}(\varphi) \), then the Hamiltonian of the circuit is given explicitly by

\[
H(\varphi, v) = \frac{\dot{q}(\varphi)}{C} + v. \tag{42}
\]

There is a continuum of equilibrium points defined by \( v = 0 \), independent of the initial flux \( \varphi = \varphi_0 \). Hence,

\[
v(t) > 0, \quad \text{if } v(0) > 0; \tag{43}
\]

and

\[
v(t) < 0, \quad \text{if } v(0) < 0; \tag{44}
\]

for \( 0 < t < \infty \). Moreover, a short-circuited memristor retains its initial flux for all times, and is therefore a nonvolatile memory.

Note that Properties 2 and 3 are duals, and \( \dot{\varphi}(q) = \int M(q) dq \) and \( \dot{q}(\varphi) = \int W(\varphi) d\varphi \), where the two functions \( M(q) \) and \( W(\varphi) \) are the memristance and memductance of the memristor, respectively.
Fig. 19. Numerical solutions of $x(t), i(t), H_x(t), H_i(t), H(t)$ and $E(t)$ of Eq. (28) as a function of time $t$, initial conditions $x(0) = -0.8, i(0) = 4$.

Fig. 20. Memristor-inductor circuit with an ideal charge-controlled memristor.

Fig. 21. Memristor-capacitor circuit with an ideal flux-controlled memristor.
3. 2-Element Memristive Circuit with Sinusoidal Forcing

Let us connect a voltage source $v_s(t) = \gamma \cos \omega t$ in series with the circuit in Fig. 1, as shown in Fig. 22. Applying Kirchhoff's voltage and current laws to this circuit, we obtain the following differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= f(i)g(x), \\
L \frac{di}{dt} &= -M(x)i + \gamma \cos \omega t,
\end{aligned}$$

If we use the Hamiltonian of Eq. (2), namely,

$$H(x,i) = \int \frac{M(x)}{L} \, dx + \int \frac{f(i)}{i} \, di,$$

Eq. (45) can be written as

$$\begin{aligned}
\frac{1}{Q} \left( \frac{dx}{dt} \right) &= \frac{\partial H}{\partial i}, \\
\frac{1}{Q} \left( \frac{di}{dt} \right) &= -\frac{\partial H}{\partial x} + \frac{\gamma \cos \omega t}{LQ},
\end{aligned}$$

where $Q(x,i) = g(x)i \neq 0$. Note that $H(x,i)$ is not conserved in Eq. (47).

We next show some examples of Eq. (45). Using the functions from Example 1, Eq. (45) becomes

$$\begin{aligned}
\frac{dx}{dt} &= \alpha (i^2 - 1), \\
\frac{di}{dt} &= -(x + \beta)i + \gamma \cos \omega t,
\end{aligned}$$

where $\alpha$, $\beta$, $\gamma$ and $\omega$ denote some constants.

![Fig. 22. 2-element memristive device circuit driven by a periodic voltage source $v_s(t)$.](image1)

![Fig. 23. (a) Trajectory of Eq. (48) and (b) its Poincaré map. Initial conditions: $x(0) = 0.1$, $i(0) = 0.1$. Parameters: $\alpha = 1$, $\beta = 0$, $\gamma = 0.7$, $\omega = 1$.](image2)
Fig. 24. Numerical solutions of \( x(t) \) and \( i(t) \) of Eq. (48) as a function of time \( t \) with initial conditions \( x(0) = 0.1, i(0) = 0.1 \), parameters \( \alpha = 1, \beta = 0, \gamma = 0.7, \omega = 1 \) and \( t \in [1000, 1200] \).

Similarly, using the functions from Example 2, Eq. (45) becomes

\[
\begin{align*}
\text{forced two-dimensional system B} & \\
\frac{dx}{dt} &= -(i + \beta)x, \\
\frac{di}{dt} &= -\alpha(x^2 - 1) + \gamma \cos \omega t,
\end{align*}
\]

(49)

where \( \alpha, \beta, \gamma \) and \( \omega \) denote some constants.

Using the functions from Example 3, Eq. (45) becomes

\[
\begin{align*}
\text{forced two-dimensional system C} & \\
\frac{dx}{dt} &= i, \\
\frac{di}{dt} &= -(x^2 + 1)i + \gamma \cos \omega t,
\end{align*}
\]

(50)

where \( \gamma \) and \( \omega \) denote some constants.

The trajectories of Eqs. (48) and (49) and their Poincaré maps, as well as the state \( x(t) \) and current \( i(t) \) are shown in Figs. 23–26 as a function of time \( t \). In Eqs. (48) and (49), we choose the parameters such that they exhibit complex Poincaré maps. In our computer study, Eq. (50) has many periodic

Fig. 25. (a) Trajectory of Eq. (49), and (b) its Poincaré map. Initial conditions: \( x(0) = 0.86, i(0) = -5 \). Parameters: \( \alpha = 1, \beta = 1, \gamma = 0.02, \omega = 1 \).
Fig. 26. Numerical solutions of $x(t)$ and $i(t)$ of Eq. (2) as a function of time $t$ with initial conditions $x(0) = 0.86$, $i(0) = -5$, parameters $\alpha = 1$, $\beta = 1$, $\gamma = 0.02$, $\omega = 1$ and $t \in [1000, 1200]$.

Fig. 27. Two typical trajectories of Eq. (50). The initial transient segment is drawn in red (a). Initial conditions: $x(0) = 0.1$, $i(0) = 0.1$ (a) and $x(0) = -0.5$, $i(0) = -0.5$ (b).

Fig. 28. Numerical solutions of $x(t)$ and $i(t)$ of Eq. (50) as a function of time $t$, with initial conditions (a) and (b) $x(0) = 0.1$, $i(0) = 0.1$, (c) and (d) $x(0) = -0.5$, $i(0) = -0.5$ and $t \in [1000, 1050]$. 
orbits. Only two examples are shown in Figs. 27 and 28. The following parameters are used in our computer studies:

- Equation (48): $\alpha = 1, \beta = 0, \gamma = 0.7, \omega = 1$.
- Equation (49): $\alpha = 1, \beta = 1, \gamma = 0.02, \omega = 1$.
- Equation (50): $\gamma = 1, \omega = 0.5$.

4. 3-Element Memristive Circuits

Consider the circuit in Fig. 29, which consists of three elements: a passive linear inductor, a passive linear capacitor, and an active memristive device described by Eq. (1). Applying Kirchhoff's current and voltage laws to this circuit, we obtain the following set of three differential equations

\[
\begin{align*}
\frac{dx}{dt} &= f(i)g(x), \\
L \frac{di}{dt} &= v - M(x)i, \\
C \frac{dv}{dt} &= -i.
\end{align*}
\]

By setting $\omega = 1/\sqrt{LC}$ and by changing the variables

\[
\begin{align*}
y &= \frac{i}{\omega}, \\
z &= -Cv,
\end{align*}
\]

we obtain

\[
\begin{align*}
\frac{dx}{dt} &= f(\omega y)g(x), \\
\frac{dy}{dt} &= -\omega z - \frac{M(x)y}{L}, \\
\frac{dz}{dt} &= \omega y.
\end{align*}
\]

If we define

\[
H_1(x, y) = \int \frac{M(x)}{L} g(x) dx + \int \frac{f(\omega y)}{y} dy,
\]
and

\[
H_2(y, z) = \frac{y^2 + z^2}{2},
\]

we can recast Eq. (53) as follow:

\[
\begin{align*}
\frac{dx}{dt} &= \mu \frac{\partial H_1(x, y)}{\partial y}, \\
\frac{dy}{dt} &= -\mu \frac{\partial H_1(x, y)}{\partial x} - \omega \frac{\partial H_2(y, z)}{\partial z}, \\
\frac{dz}{dt} &= \omega \frac{\partial H_2(y, z)}{\partial y},
\end{align*}
\]

where $\mu = yg(x)$. It follows from Eqs. (52) and (53) that

\[
\begin{align*}
\frac{dH_1(x, y)}{dt} &= 0, \quad \text{if } \omega = 0, \\
\frac{dH_2(y, z)}{dt} &= 0, \quad \text{if } \mu = 0.
\end{align*}
\]

Let us consider next some examples of this 3-element memristive circuit.

4.1. *Nosé–Hoover oscillator*

Choose

\[
\begin{align*}
f(\omega y) &= \alpha(y^2 - 1), \\
g(x) &= 1, \\
\frac{M(x)}{L} &= x.
\end{align*}
\]

Substituting Eq. (58) into Eq. (53), we obtain the following Nosé–Hoover oscillator equations:
[Posch et al., 1986]

\[
\begin{aligned}
\text{Nosé–Hoover oscillator} & \\
\frac{dx}{dt} & = \alpha (y^2 - 1), \\
\frac{dy}{dt} & = -\omega z - xy, \\
\frac{dz}{dt} & = \omega y,
\end{aligned}
\]

where \(\alpha\) denotes a constant and \(\omega = 1\). The attractor of Eq. (59) is shown in Fig. 30. The Nosé–Hoover oscillator equations can be derived from the Nose “many-body” Hamiltonian [Posch et al., 1986], which can be transcribed to

\[
H = \frac{x^2}{2\alpha} + \frac{y^2}{2} + \frac{z^2}{2} + \int_{0}^{t} x(s)ds,
\]

for more details, see [Posch et al., 1986] and Appendix B).

Substituting

\[
\omega = \frac{\lambda}{1-\lambda}, \quad t = (1-\lambda)\tau,
\]

into Eq. (59), we obtain

\[
\begin{aligned}
\text{third-order system with parameter } \lambda & \\
\frac{dx}{d\tau} & = (1-\lambda)\alpha (y^2 - 1), \\
\frac{dy}{d\tau} & = -\lambda z - (1-\lambda)xy, \\
\frac{dz}{d\tau} & = \lambda y,
\end{aligned}
\]

where the parameter \(\lambda\) defines an artificial homotopy parameter.\(^7\)\(^8\)

Equation (62) can be shown to have the following integral

\[
I = \frac{x^2}{2\alpha} + \frac{y^2}{2} + \frac{z^2}{2} + (1-\lambda)\int_{0}^{t} x(s)ds.
\]

Let us study the dynamics of Eq. (62) over the parameter range \(\lambda \in [0, 1]\).

\(^7\)In the case of Nosé–Hoover oscillator, we set \(\omega = 1\). In order to treat \(\omega\) as a variable parameter, we made the change of variables \(\omega = \lambda/1-\lambda\) and \(\lambda\) is treated as a variable parameter.

\(^8\)The vector field associated with Eq. (59) in \(\mathbb{R}^3\) can be continuously deformed by tuning the parameter \(\lambda\), over a wide range of parameter values, except at some critical parameters where bifurcations occur.
Case 1. $\lambda = 0$

In this case, Eq. (62) reduces to

$$\begin{align*}
\frac{dx}{d\tau} &= \alpha(y^2 - 1), \\
\frac{dy}{d\tau} &= -xy, \\
\frac{dz}{d\tau} &= 0,
\end{align*}$$

(64)

which is equivalent to the dynamics of the 2-element memristive circuit in Fig. 1, represented by Eq. (14) with $\beta = 0$.

Case 2. $\lambda = 1/2$

In this case, we obtain

$$\begin{align*}
\frac{dx}{ds} &= \alpha(y^2 - 1), \\
\frac{dy}{ds} &= -z - xy, \\
\frac{dz}{ds} &= y.
\end{align*}$$

(65)

This equation is equivalent to the Nosé–Hoover oscillator equations ($\omega = 1$), where $\tau = s/2$.

Case 3. $\lambda = 1$

In this case, Eq. (62) reduces to

$$\begin{align*}
\frac{dx}{d\tau} &= 0, \\
\frac{dy}{d\tau} &= -z, \\
\frac{dz}{d\tau} &= y.
\end{align*}$$

(66)

This equation is equivalent to the dynamics of the linear LC circuit shown in Fig. 31.

The bifurcation of trajectories of Eq. (62) with $\alpha = 1$ and $\lambda \in [0, 1]$ is shown in Fig. 32. Observe that this 3-element memristive circuit exhibits rather interesting bifurcation phenomena. The state $x(t)$, $y(t)$, and $z(t)$ of Eq. (62) with $\lambda = 0.5$ and 0.9 are shown as a function of time $t$ in Figs. 33 and 34, respectively.

4.2. 3-Element oscillator

Choose

$$\begin{align*}
f(\omega y) &= y + \beta, \\
g(x) &= -x, \\
\frac{M(x)}{L} &= \alpha(x^2 - \gamma),
\end{align*}$$

(67)

where $\alpha$, $\beta$ and $\gamma$ denote some constants. Substituting Eq. (67) into Eq. (53), we obtain

$$\begin{align*}
\begin{cases}
\frac{dx}{dt} = -x(y + \beta), \\
\frac{dy}{dt} = -\omega z - \alpha(x^2 - \gamma)y, \\
\frac{dz}{dt} = \omega y.
\end{cases}
\end{align*}$$

(68)

The trajectory of Eq. (68) with $\alpha = 1, \beta = 1, \gamma = 0.1$ and $\omega = 1$ is shown in Fig. 35. Observe that it exhibits rather complex behaviors.

Equation (68) can be recast into

$$\begin{align*}
\begin{cases}
\frac{dx}{d\tau} = -(1 - \lambda)x(y + \beta), \\
\frac{dy}{d\tau} = -\lambda z - (1 - \lambda)\alpha(x^2 - \gamma)y, \\
\frac{dz}{d\tau} = \lambda y,
\end{cases}
\end{align*}$$

(69)

where

$$\omega = \frac{\lambda}{1 - \lambda}, \quad \tau = \frac{t}{1 - \lambda}.$$  

(70)

Let us examine the dynamics of Eq. (69) over the parameter range $\lambda \in [0, 1]$. 

---

Fig. 31. A 2-element circuit consisting of an inductor and a capacitor, whose inductance and capacitance are 1H and 1F, respectively.
Fig. 32. Bifurcation of trajectories of Eq. (62) for the parameter $\lambda \in [0, 1]$. Initial conditions: $x(0) = 4.1, y(0) = 0.7, z(0) = 5$. The trajectories are coded using the color code for $z(t)$. As $z(t)$ varies from minimum to maximum, the color runs through red, yellow, green, cyan, blue, magenta, and back to red again.
Fig. 33. Numerical solutions of $x(t)$, $y(t)$ and $z(t)$ of Eq. (62) as a function of time $t$ with initial conditions $x(0) = 4.1, y(0) = 0.7, z(0) = 5$, parameter $\lambda = 0.5$ and $t \in [1000, 1300]$.

- **Case 1.** $\lambda = 0$
  In this case, Eq. (69) reduces to
  \[
  \begin{align*}
  \frac{dx}{d\tau} &= -x(y + \beta), \\
  \frac{dy}{d\tau} &= -\alpha(x^2 - \gamma)y, \\
  \frac{dz}{d\tau} &= 0.
  \end{align*}
  \]  \hspace{1cm} (71)

  This equation is equivalent to the dynamics of the 2-element memristive circuit in Fig. 1, represented by Eq. (21).

- **Case 2.** $\lambda = 1/2$
  \[
  \begin{align*}
  \frac{dx}{ds} &= -x(y + \beta), \\
  \frac{dy}{ds} &= -z - \alpha(x^2 - \gamma)y, \\
  \frac{dz}{ds} &= y.
  \end{align*}
  \]  \hspace{1cm} (72)

This equation is similar to the simplest chaotic circuit equation in [Muthuswamy & Chua, 2010], where $\tau = s/2$. An attractor of Eq. (72) is shown in Fig. 35.

- **Case 3.** $\lambda = 1$
  Equation (69) reduces to
  \[
  \begin{align*}
  \frac{dx}{d\tau} &= 0, \\
  \frac{dy}{d\tau} &= -z, \\
  \frac{dz}{d\tau} &= y.
  \end{align*}
  \]  \hspace{1cm} (73)

  This equation is equivalent to the dynamics of the 2-element circuit in Fig. 31.

The bifurcation of trajectories of Eq. (69) with $\alpha = 1, \beta = 1, \gamma = 0.1$ and $\lambda \in [0, 1]$ is shown in Fig. 36. Observe that Eq. (69) exhibits interesting
Fig. 35. Trajectory of Eq. (69), viewed from two different perspectives. Initial conditions: $x(0) = 1, y(0) = 1, z(0) = 1$. Parameters: $\alpha = 1, \beta = 1, \gamma = 0.1, \omega = 1$.

bifurcation phenomena. The states $x(t)$, $y(t)$ and $z(t)$ of Eq. (69) with $\lambda = 0.1, 0.5$ are shown as a function of time $t$ in Figs. 37 and 38.

5. 2N-Element Memristive Circuits

Consider the $N$-particle Hamiltonian which is the sum of kinetic and potential energies denoted $T$ and $V$, respectively:

$$H = T + V,$$

$$T = \sum_{k=1}^{N} \frac{p_{k}^2}{2m_{k}},$$

$$V = V(q_1, q_2, \ldots, q_N),$$

(74)

where $q_i$ is the position of $i$th particle whose mass is $m_i$, and $p_i$ is the momentum $p_i = m_i dq_i / dt$. Then the Hamiltonian equation is written as follows:

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} = \frac{p_k}{m_k},$$

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} = -\frac{\partial V}{\partial q_k},$$

(75)

Many dynamical systems can be described by Eqs. (74) and (75) [Meiss, 2007].

Substituting

$$m_k = L_k,$$

$$p_k = L_k \ln |i_k|,$$

$$q_k = x_k,$$

(76)

into Eqs. (74) and (75), we obtain

$$\tilde{H} = \tilde{T} + \tilde{V},$$

$$\tilde{T} = \sum_{i=1}^{N} \left( \frac{\ln |i_k|^2}{2} \right),$$

$$\tilde{V} = V(x_1, x_2, \ldots, x_N),$$

(77)

and

2N-element memristive circuit

$$\frac{dx_k}{dt} = \ln |i_k|,$$

$$L_k \frac{di_k}{dt} = -\left( \frac{\partial \tilde{V}}{\partial x_k} \right) i_k,$$

(78)
Fig. 36. Bifurcation of trajectories of Eq. (69) for the parameter $\lambda \in [0, 1]$. Initial conditions for $\lambda \neq 0$: $x(0) = 0.5, y(0) = 0.5, z(0) = 0.5$. Initial conditions for $\lambda = 0$: $x(0) = 1.5, y(0) = -1.5, z(0) = 0.5$. The trajectories are coded using the same color code for $z(t)$. As $z(t)$ varies from minimum to maximum, the color runs through red, yellow, green, cyan, blue, magenta, and back to red again.
Fig. 37. Numerical solutions of \( x(t) \), \( y(t) \) and \( z(t) \) of Eq. (69) as a function of time \( t \) with initial conditions \( x(0) = 0.5 \), \( y(0) = 0.5 \), \( z(0) = 0.5 \), parameter \( \lambda = 0.1 \) and \( t \in [1000, 1500] \).

Fig. 38. Numerical solutions of \( x(t) \), \( y(t) \) and \( z(t) \) of Eq. (69) as a function of time \( t \) with initial conditions \( x(0) = 0.5 \), \( y(0) = 0.5 \), \( z(0) = 0.5 \), parameter \( \lambda = 0.5 \) and \( t \in [1000, 1300] \).

Fig. 39. A \( 2N \)-element Hamiltonian circuit, which consists of \( N \) inductors with the inductance \( L_k \) (\( k = 1, 2, \ldots, N \)) and \( N \) memristive devices described by \( v_k = M_k(x_1, x_2, \ldots, x_N) \) \( i_k = ((\partial V(x_1, x_2, \ldots, x_N))/\partial x_k) i_k \), \( dx_k/dt = \ln |i_k| \) (\( k = 1, 2, \ldots, N \)), where \( M_k(x_1, x_2, \ldots, x_N) \) denotes the memristance of the \( k \)th memristive device and \( V(x_1, x_2, \ldots, x_N) \) denotes the potential energy of the \( N \)-particle Hamiltonian. Even though the memristive devices appear to be disconnected, their dynamics are coupled via the memristance equation involving the same state variables \( (x_1, x_2, \ldots, x_N) \).
respectively, where \( i_k \neq 0 \). Equation (78) has the solution \( H = H_0 \) \((H_0 \text{ is any constant})\), and it can be recast into Eq. (75) by changing the variables

\[
\begin{align*}
    i_k &= e^{v_k}, \\
    x_k &= q_k, \\
    L_k &= m_k.
\end{align*}
\] (79)

Consider the current-controlled memristive device described by

\[
\begin{align*}
    \frac{dx_k}{dt} &= \ln |i_k|, \\
    v_k &= M_k(x_1, x_2, \ldots, x_N)i_k,
\end{align*}
\] (80)

where \( v_k \) and \( i_k \) denote the terminal voltage and current of the \( k \)th memristive device, whose states are coupled among the \( N \) memristive devices, whose memristance is described by

\[
M_k(x_1, x_2, \ldots, x_N) = \frac{\partial V(x_1, x_2, \ldots, x_N)}{\partial x_k}.
\] (81)

Then, Eq. (78) can be realized by \( 2N \)-element memristive circuit shown in Fig. 39.

**Example.** Choose

\[
\begin{align*}
    T(p_1, p_2) &= \frac{p_1^2 + p_2^2}{2}, \\
    V(q_1, q_2) &= \frac{q_1^2 + q_2^2}{2} + q_1^2 q_2 - \frac{1}{3} q_2^3, \\
    p_k &= \ln |i_k|, \\
    m_k &= 1,
\end{align*}
\] (82)

where \( k = 1, 2 \). Then Eqs. (75) and (78) are recast respectively as

\[
\begin{align*}
    \frac{dq_1}{dt} &= p_1, \\
    \frac{dq_2}{dt} &= p_2, \\
    \frac{dp_1}{dt} &= -q_1 - 2q_1 q_2, \\
    \frac{dp_2}{dt} &= -q_2 - q_1^2 + q_2^2.
\end{align*}
\] (83)

\[
\begin{align*}
    \frac{dx_1}{dt} &= \ln |i_1|, \\
    \frac{dx_2}{dt} &= \ln |i_2|, \\
    \frac{di_1}{dt} &= (-x_1 - 2x_1 x_2)i_1, \\
    \frac{di_2}{dt} &= (-x_2 - x_1^2 + x_2^2)i_2.
\end{align*}
\] (84)

Equation (83) is a nonlinear nonintegrable Hamiltonian equation, called Hénon and Heiles equation [Hénon & Heiles, 1964].

The Hamiltonians of Eqs. (75) and (78) are described by

\[
H = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^2 + q_2^2}{2} + q_1^2 q_2 - \frac{1}{3} q_2^3,
\] (85)

\[
\tilde{H} = \frac{(\ln |i_1|)^2 + (\ln |i_2|)^2 + x_1^2 + x_2^2}{2} + x_1^2 x_2 - \frac{1}{3} x_2^3,
\] (86)

respectively. Trajectories of Eqs. (75) and (78) move on surfaces of these Hamiltonians. We show the Poincaré sections\(^9\) of Eqs. (75)–(78) in Figs. 40 and 41. Observe that they have some elliptic islands and a chaotic region.

\(^9\)A Poincaré section (surface of section) is a method of displaying a trajectory in \( n \)-dimensional phase space in an \( n-1 \) dimensional space. It is obtained by setting one element constant and sampling the values of the other elements each time the selected element has the desired value Hénon and Heiles [1964].
Fig. 40. Poincaré sections of the Hénon and Heiles equation (83) starting from 50 random initial conditions with $H = 0.125$ (left) and $H = 0.11$ (right). Points $(p_2(t), q_2(t))$ are successively plotted when $q_1(t) = 0$. In the top figures, all trajectories are plotted in red. In the bottom figures, different color represents different trajectory.
Fig. 41. Poincaré sections of the memristor Hénon and Heiles equation (84) starting from 50 random initial conditions with $\dot{H} = 0.125$ (left) and $\dot{H} = 0.11$ (right). Points $(i_2(t), x_2(t))$ are successively plotted when $x_1(t) = 0$. In the top figures, all trajectories are plotted in red. In the bottom figures, different color represents different trajectory.
6. Concluding Remarks

We have studied several memristor Hamiltonian circuits, and showed that the memristive device exhibits many nonintuitive and interesting dynamic behaviors.

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Appendix A

Hamiltonian Equation and Pfaff’s Equation

The Lagrange equation can be written as [Andronov et al., 1987]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \]  
(A.1)

where \( L(q, \dot{q}) \) denotes a single valued function of the coordinate \( q \) and the velocity \( \dot{q} \equiv dq/dt \), called the Lagrangian function. The Lagrange equation can be also written in the form

\[ \dot{p} = \frac{\partial L}{\partial q} \]  
(A.2)

where \( p = \partial L/\partial \dot{q} \) denotes the momentum.

Hamiltonian equation can be obtained by the variational principle [Andronov et al., 1987]

\[ \delta \int_{t_0}^{t_1} L(q, \dot{q})dt = \delta \int_{t_0}^{t_1} \{ p\dot{q} - H(p, q) \} dt = 0, \]  
(A.3)

where \( L(q, \dot{q}) \) denotes a Lagrangean function, \( H(p, q) = p\dot{q} - L \) denotes the Hamiltonian, and \( \delta \) denotes an arbitrary smooth variation of the path subject to the boundary conditions \( \delta q(t_0) = \delta q(t_1) = 0 \). Equation (A.3) can be written as

\[ \int_{t_0}^{t_1} \left( \delta \dot{p} + \delta \dot{q} - \frac{\partial H}{\partial q} \delta q - \frac{\partial H}{\partial p} \delta p \right) dt = 0. \]  
(A.4)

Applying the integration by parts rule to the second term of Eq. (A.4), we obtain

\[ \int_{t_0}^{t_1} \delta \dot{q} dt = \left[ \delta \dot{q} \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p} \delta q dt = - \int_{t_0}^{t_1} \dot{p} \delta q dt. \]  
(A.5)

Thus, Eq. (A.4) can be described by

\[ \int_{t_0}^{t_1} \left\{ \left( \dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left( \dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right\} dt = 0. \]  
(A.6)

From Eq. (A.6), we obtain the Hamiltonian equation

\[ \dot{q} = \frac{\partial H}{\partial p}, \]  
(A.7)

\[ \dot{p} = - \frac{\partial H}{\partial q}. \]

Observe that the Hamiltonian equation has the integral

\[ H(p, q) = h, \]  
(A.8)

where \( h \) is any constant.

Consider now the more general variational form [Andronov et al., 1987]

\[ \delta \int_{t_0}^{t_1} (X\dot{x} + Y\dot{y} - F) dt = 0, \]  
(A.9)
where $X, Y$ and $F$ are single valued functions of $x$ and $y$. Since
\[
\delta \int_{t_0}^{t_1} (X\dot{x}) dt = \int_{t_0}^{t_1} (\delta X\dot{x} + X\delta \dot{x}) dt
\]
\[
= \int_{t_0}^{t_1} (\delta X\dot{x} - \dot{x}\delta X) dt + X\delta \dot{x} |_{t_0}^{t_1}
\]
\[
= \int_{t_0}^{t_1} (\delta X\dot{x} - \delta x\dot{X}) dt
\]
\[
= \int_{t_0}^{t_1} \left\{ \left( \frac{\partial X}{\partial x} \delta x + \frac{\partial X}{\partial y} \delta y \right) \dot{x}
\]
\[
- \left( \frac{\partial X}{\partial x} \dot{x} + \frac{\partial X}{\partial y} \dot{y} \right) \right\} dt
\]
\[
= \int_{t_0}^{t_1} \left\{ \left( \frac{\partial X}{\partial y} \dot{y} - \frac{\partial X}{\partial x} \dot{x} \right) \right\} dt.
\]  
\[\text{(A.10)}\]

\[
\delta \int_{t_0}^{t_1} (Y\dot{y}) dt = \int_{t_0}^{t_1} (\delta Y\dot{y} + Y\delta \dot{y}) dt
\]
\[
= \int_{t_0}^{t_1} \left\{ \left( \frac{\partial Y}{\partial x} \dot{x} + \frac{\partial Y}{\partial y} \dot{y} \right) \right\} dt
\]
\[
= \int_{t_0}^{t_1} \left\{ \left( \frac{\partial Y}{\partial x} \dot{x} - \frac{\partial Y}{\partial x} \dot{x} \right) \right\} dt.
\]  
\[\text{(A.11)}\]

\[
\delta \int_{t_0}^{t_1} F dt = \int_{t_0}^{t_1} \left( \frac{\partial F}{\partial x} \delta x - \frac{\partial F}{\partial y} \delta y \right) dt
\]
\[
= \int_{t_0}^{t_1} \left\{ \left( Q(x,y)\dot{x} - \frac{\partial F}{\partial y} \right) \right\} \delta y
\]
\[
- \left( Q(x,y)\dot{y} + \frac{\partial F}{\partial x} \right) \delta x \right\} dt = 0,
\]  
\[\text{(A.13)}\]

where $Q(x,y) = \partial X / \partial y - \partial Y / \partial x$. Hence, we obtain the Pfaff's equation [Andronov et al., 1987]
\[
Q(x,y)\dot{x} = \frac{\partial F}{\partial y},
\]
\[
Q(x,y)\dot{y} = - \frac{\partial F}{\partial x},
\]  
\[\text{(A.14)}\]

which is the general form of equations describing a conservative system. Equation (A.14) has the solution $F(x,y) = C$ ($C$ is any constant), since
\[
\frac{dF(x,y)}{dt} = \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial y} \dot{y}
\]
\[
= \{-Q(x,y)\dot{y}\} \dot{x} + \{Q(x,y)\dot{x}\} \dot{y} = 0.
\]  
\[\text{(A.15)}\]

If $Q(x,y) > 0$, Eq. (A.14) can be transformed into the Hamiltonian system. That is, by transforming the independent variable $t$ by means of the formula $d\tau = dt / Q(x,y)$, we obtain
\[
\frac{dx}{d\tau} = \frac{\partial F}{\partial y},
\]
\[
\frac{dy}{d\tau} = - \frac{\partial F}{\partial x}.
\]  
\[\text{(A.16)}\]

In this transformation, we multiply the velocity at the point $(x, y)$ by the value of the function $Q(x, y)$ at this point, however, it does not alter the trajectories [Nemytskii & Stepanov, 1989]. If $Q(x, y) < 0$, the time orientation of orbits is reversed. Note that Eqs. (A.14) and (A.16) are recast into the same form
\[
\frac{dy}{dx} = - \frac{\partial F}{\partial x},
\]  
\[\text{(A.17)}\]

if $Q(x,y) \neq 0$ and $\partial F / \partial y \neq 0$.

**Appendix B**

**Nosé–Hoover Oscillator Equation**

Consider the following Hamiltonian from [Posch et al., 1986]:
\[
H(P, Q, P_s, s) = G(Q) + F\left(\frac{P}{s}\right) + \ln s + \frac{\alpha P_s^2}{2},
\]  
\[\text{(B.1)}\]

where $F(\cdot)$ and $G(\cdot)$ are scalar functions. Here, $Q$ denotes the position, $P$ denotes the momentum, $s$ denotes the time scale variable (or effective mass variable), and $P_s$ denotes the conjugate momentum. The equations of motion follow from Eq. (B.1):
\[
\begin{align*}
\frac{dQ}{dt} &= \frac{\partial H}{\partial P} = F' \left( \frac{P}{s} \right) \frac{1}{s}, \\
\frac{dP}{dt} &= -\frac{\partial H}{\partial Q} = -G'(Q), \\
\frac{ds}{dt} &= \frac{\partial H}{\partial P_s} = \alpha P_s, \\
\frac{dP_s}{dt} &= -\frac{\partial H}{\partial s} = F' \left( \frac{P}{s} \right) \frac{P}{s^2} - \frac{1}{s}.
\end{align*}
\]

After time scaling by \( d\tau = dt/s \), we get
\[
\begin{align*}
\frac{dQ}{d\tau} &= F' \left( \frac{P}{s} \right), \\
\frac{dP}{d\tau} &= -sG'(Q), \\
\frac{ds}{d\tau} &= \alpha sP_s, \\
\frac{dP_s}{d\tau} &= F' \left( \frac{P}{s} \right) \frac{P}{s} - 1.
\end{align*}
\]

Changing the variables
\[
\begin{align*}
q &= Q, \\
p &= \frac{P}{s}, \\
\zeta &= \alpha P_s,
\end{align*}
\]
we obtain
\[
\begin{align*}
\frac{dq}{d\tau} &= \frac{dQ}{d\tau} = F' \left( \frac{P}{s} \right) = F'(p), \\
\frac{dp}{d\tau} &= \frac{dP}{d\tau} = \frac{1}{s} \frac{dP}{d\tau} - \frac{P}{s^2} \frac{ds}{d\tau} \\
&= \frac{1}{s} (-sG'(q)) - \left( \frac{P}{s^2} \right) \alpha sP_s \\
&= -G'(q) - \alpha P_s \left( \frac{P}{s} \right) \\
&= -G'(q) - \alpha P_s \left( \frac{P}{s} \right) \\
&= -G'(q) - \alpha P_s \left( \frac{P}{s} \right) \\
&= -G'(q) - \zeta p, \\
\frac{d\zeta}{d\tau} &= \alpha \frac{d\zeta}{d\tau} = \alpha \left( F' \left( \frac{P}{s} \right) \frac{P}{s} - 1 \right) \\
&= \alpha (pF'(p) - 1).
\end{align*}
\]

Hence, we have a set of differential equations for the variables \((q,p,\zeta)\)

Furthermore, from the third equation in Eq. (B.3), we obtain the following relation
\[
\frac{1}{s} \left( \frac{ds}{d\tau} \right) = \frac{d\ln s}{d\tau} = \alpha P_s = \zeta.
\]

Integrating the equation
\[
\frac{1}{s} \left( \frac{ds}{d\tau} \right) = \zeta,
\]
we get
\[
\ln s(\tau) = \int^\tau \zeta(t)dt.
\]

Hence, the Hamiltonian (B.1) can be written as
\[
H = G(Q) + F \left( \frac{P}{s} \right) + \ln s + \alpha P_s^2
\]
\[
= G(q) + F(p) + \ln s + \frac{\zeta^2}{2}
\]
\[
= G(q) + F(p) + \int^\tau \zeta(t')dt' + \frac{\zeta^2}{2}.
\]
Assume that
\[
F(p) = \frac{p^2}{2},
\]
\[
G(q) = \frac{q^2}{2}
\]
Then, from Eqs. (B.4) and (B.10), we obtain the Nosé–Hoover oscillator
\[
\begin{align*}
\frac{dq}{d\tau} &= p, \\
\frac{dp}{d\tau} &= -q - \zeta p, \\
\frac{d\zeta}{d\tau} &= \alpha (p^2 - 1),
\end{align*}
\]
and the Hamiltonian
\[
H = \frac{p^2}{2} + \frac{q^2}{2} + \frac{\zeta^2}{2} + \int^\tau \zeta(t')dt',
\]
respectively.