Distributed Algorithms for Minimum Cost Multicast with Network Coding

Yufang Xi and Edmund M. Yeh
Department of Electrical Engineering
Yale University
New Haven, CT 06520, USA
{yufang.xi,edmund.yeh}@yale.edu

Abstract

We consider the problem of finding the minimum-cost multicast scheme for a single session with elastic rate demand based on the network coding approach. It is shown that solving for the optimal coding subgraphs in network coding is equivalent to finding the optimal routing scheme in a multicommodity flow problem. We design a set of node-based distributed gradient projection algorithms consisting of joint congestion control/routing at the source node and “virtual” routing at intermediate nodes. With appropriately chosen parameters, we show that the distributed algorithms converge to the optimal configuration from all initial conditions.

I. INTRODUCTION

Routing has long been an important technique in optimizing data transmissions for a communication network. In conventional optimal routing problems for wired networks, each node functions as a switch for passing data streams. The node relays the data it receives, but makes no change to the data content. Analytically, the situation can be treated as a Multi-commodity Flow Problem (MFP) [1], [2], where data streams are treated as commodities identified by their different destinations. As they are routed among nodes, all commodities maintain their distinctness. Node-based distributed algorithms are presented in [1], [2] to achieve routing patterns that minimize overall network cost. To prevent excessive data input flows from overloading capacitated networks, congestion control algorithms used at source nodes are developed [3], [4]. The combination of congestion control and routing [5]–[7] provides an overall optimal solution to the MFP described above.

The recent breakthrough in network coding [8], [9] extends the functionality of network nodes to performing algebraic operations on received data. In general, network coding techniques improve network throughput [8], network robustness [10], and the efficiency of network resource allocation [11], over those achievable by pure routing. The advantage of network coding is most pronounced in establishing multicast connections. Li et al. [12] prove that linear coding suffices to obtain the optimal throughput of a multicast session, achieving the fundamental max-flow-min-cut upper bound. Decentralized random linear coding schemes are proposed in [13], [14], thus rendering network coding applicable to real networks.

The problem of finding the minimum-cost multicast scheme using a network coding approach is addressed in [15]. It is shown [15] that the solution of this problem can be decomposed into two parts: finding the minimum-cost coding subgraphs and designing the code applied over the optimal subgraphs. A distributed solution for the second part was provided in [13]. To solve the first part, the work in [11] proposes a distributed algorithm for finding the optimal coding subgraphs via a primal-dual approach. This approach, however, is

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complicated by the fact that distributed evaluation of the dual function is in itself a complex problem. Motivated by this, we design a set of node-based primal gradient projection algorithms that iteratively find the minimum-cost coding subgraphs. Furthermore, we explicitly specify the proper scaling matrices and step sizes for each of the algorithms and show that these parameters can be calculated efficiently in a distributed way. Interestingly, although network coding represents a decidedly different network management approach than routing, solving for the optimal coding subgraphs in network coding intrinsically resembles finding the optimal routing scheme in an MFP. In this work, we fully explore this connection and transform the subgraph searching problem into a “virtual” MFP. Furthermore, we generalize the distributed optimal routing algorithms developed in [1], [2] to design a complete set of distributed solutions for the optimal multicast problem involving both congestion control and network coding. In contrast to [2], our scheme uses a different technique for computing the scaling matrices and step sizes. This scheme allows us to guarantee the convergence of the algorithms from all initial conditions.

II. Problem Formulation

We consider finding the minimum-cost network coding subgraphs for a single multicast session with elastic rate demand. This procedure, followed by a network code designed specifically for the derived subgraphs, provides the optimal network configuration for the multicast session. Let the network supporting the multicast session be modelled by a directed and connected graph $G = (N, E)$. Specifically, denote the unique source node in $N$ by $s$, and use $W \subseteq N \setminus \{s\}$ to represent the set of multiple destination nodes. For each $w \in W$, we say $(s, w)$ is the source-destination pair of virtual session $w$.

To measure the optimality of a multicast scheme, we first associate a utility function $U(\cdot)$ with the multicast session. Assume the session’s maximal transmission rate is $R$ bits/sec, i.e. no more utility is gained by transmitting at rate $r \geq R$. As a function of the admitted rate $r$, $U(r)$ is strictly increasing and concave on its domain $[0, R]$. For analytical purposes, assume $U(r)$ is twice continuously differentiable.

We adopt a flow model to analyze the transmission of the multicast session’s data traffic from the source $s$ to respective destinations $w \in W$. A cost measured by function $D_{ij}(F_{ij})$ is incurred on link $(i, j)$ when it transports the session’s traffic at rate $F_{ij}$. We assume that the total network cost is $\sum_{(i, j)} D_{ij}(F_{ij})$. If costs also exist at nodes, they can be absorbed into the costs of the nodes’ adjacent links. Furthermore, if each link $(i, j)$ has finite capacity $C_{ij}$, we can let $D_{ij}$ implicitly impose link capacity constraint $F_{ij} \leq C_{ij}$ by defining $D_{ij}(F_{ij}) = \infty$ when $F_{ij} > C_{ij}$. In general, $D_{ij}(\cdot)$ is assumed to be convex, strictly increasing, and twice continuously differentiable on $[0, C_{ij}]$. Such link cost functions are adopted in pure routing problem for unicast networks [1], [2].

The network coding technique asymptotically provides a link flow distribution $(F_{ij})_{(i, j) \in E}$ for a multicast session with admitted rate $r$ that satisfies the following virtual flow conservation relations [8], [15]:

$$f_{ij}(w) \geq 0, \quad \forall (i, j) \in E \text{ and } w \in W, \quad (1)$$

$$\sum_{j \in O(s)} f_{sj}(w) = r \equiv t_{s}(w), \quad \forall w \in W, \quad (2)$$

$$f_{wj}(w) = 0, \quad \forall w \in W \text{ and } j \in O(w), \quad (3)$$

$$\sum_{j \in O(i)} f_{ij}(w) = \sum_{j \in T(i)} f_{ji}(w) \equiv t_{i}(w), \quad \forall w \in W \text{ and } i \in N \setminus \{s, w\}, \quad (4)$$

$$F_{ij} = \max_{w \in W} f_{ij}(w), \quad \forall (i, j) \in E, \quad (5)$$
where $\mathcal{O}(i) \triangleq \{j : (i, j) \in \mathcal{E}\}$ and $\mathcal{I}(i) \triangleq \{j : (j, i) \in \mathcal{E}\}$ represent the sets of node $i$’s next-hop and previous-hop neighbors in the network, respectively. The flow constraints in (1)-(5) essentially reflect the max-flow-min-cut bound which is achievable by network coding. The above constraints are “virtual” in the sense that the conventional flow balance equation (4) and source-sink equations (2)-(3) are with respect to flows of individual virtual sessions $(f_{ij}(w))_{(i,j)\in \mathcal{E}}$. The main difference between the optimal coding subgraph problem and the traditional optimal routing problem is that in the optimal subgraph problem, the actual link flow $F_{ij}$ is the maximum (rather than the sum) of the virtual session flows $f_{ij}(w)$ (cf. (5)). In the above equations, we use $t_i(w)$ to denote the total incoming rate of virtual session $w$ at node $i$.

The optimal multicast scheme is derived from balancing the session’s rate demand and the resulting network cost as follows:

\[
\text{maximize } \quad U(r) - \sum_{(i,j)\in \mathcal{E}} D_{ij}(F_{ij}) \\
\text{subject to } \quad 0 \leq r \leq R, \\
\quad \text{Virtual flow conservation (1)-(5).}
\]

Denote the rejected rate of the multicast session by $F_{ss'} \triangleq R - r$. Further define the “overflow” cost as $D_{ss'}(F_{ss'}) = U(R) - U(r)$, which is strictly increasing, convex, and twice continuously differentiable on $[0, R]$. Note the resemblance of $D_{ss'}$ to ordinary link cost functions. One can think of the rejected flow as being routed on a virtual overflow link [16] connecting $s$ directly to a virtual sink $s'$. Thus, the former optimization problem (6) is equivalent to the following Jointly Optimal Congestion control and Routing (JOOCR) problem:

\[
\text{minimize } \sum_{(i,j)\in \mathcal{E}} D_{ij}(F_{ij}) + D_{ss'}(F_{ss'}). 
\]

In what follows, the above objective function is denoted by $D$. The adjustment of $F_{ss'}$ corresponds to the congestion control mechanism at $s$. The flow distribution $(F_{ij})$ is determined by the coding subgraph configuration. Algorithmically, the flow variables are adjusted by the “virtual” routing functionality inside the network. An example multicast network with three destinations, one virtual sink, and one virtual overflow link is illustrated in Figure 1.

For analytical purposes, we use the approximation proposed in [11]:

\[
F_{ij} = \max_{w\in \mathcal{W}} f_{ij}(w) \approx \left( \sum_{w\in \mathcal{W}} (f_{ij}(w))^n \right)^{1/n},
\]

Later, we use “session” and “virtual session” interchangeably when this causes no confusion.
to make the actual flow $F_{ij}$ differentiable in every virtual flow variable $f_{ij}(w)$. In this case, the partial derivatives are given by

$$\frac{\partial F_{ij}}{\partial f_{ij}(w)} = \left( \frac{f_{ij}(w)}{F_{ij}} \right)^{n-1},$$

and $F_{ij}$ is strictly convex in $(f_{ij}(w))$. Approximation (10) becomes exact as $n \to \infty$. In the sequel, we assume $n$ is very large and solve JOCR problem in (9) with constraint (5) replaced by (10).

### III. NODE-BASED OPTIMIZATION VARIABLES AND OPTIMALITY CONDITIONS

We have shown that finding the optimal coding subgraphs is equivalent to solving for the minimum-cost flow distribution. The problem can therefore be tackled with an optimal routing methodology. To enable each node to independently adjust its virtual flow values, we adopt the routing variables introduced in [1]. For source node $s$, they are defined as

$$\phi_{ss'} = \frac{F_{ss'}}{R}, \quad \phi_{sj}(w) = \frac{f_{sj}(w)}{R}, \quad \forall w \in \mathcal{W} \text{ and } j \in \mathcal{O}(s),$$

and for intermediate node $i \in \mathcal{N}\{s, w\}$, they are defined as

$$\phi_{ij}(w) = \frac{f_{ij}(w)}{t_i(w)}, \quad \forall w \in \mathcal{W} \text{ and } j \in \mathcal{O}(i).$$

These newly defined variables are subject to node-based simplex constraints

$$\phi_{ss'} \geq 0, \quad \phi_{sj}(w) \geq 0, \quad \phi_{ss'} + \sum_{j \in \mathcal{O}(s)} \phi_{sj}(w) = 1, \quad \forall w \in \mathcal{W}, \quad (14)$$

$$\phi_{ij}(w) \geq 0, \quad \sum_{j \in \mathcal{O}(i)} \phi_{ij}(w) = 1, \quad \forall w \in \mathcal{W} \text{ and } i \neq s, w. \quad (15)$$

When $t_i(w) = 0$, the values of $\{\phi_{ij}(w)\}$ are immaterial. However, they are required to conform to (15) for consistency.

The first derivatives of the objective function with respect to the optimization variables are

$$\frac{\partial D}{\partial \phi_{ss'}} = R \cdot \delta \phi_{ss'}, \quad \frac{\partial D}{\partial \phi_{sj}(w)} = R \cdot \delta \phi_{sj}(w), \quad \text{and} \quad \frac{\partial D}{\partial \phi_{ij}(w)} = t_i(w) \cdot \delta \phi_{ij}(w), \quad (16)$$

where the marginal cost indicators are defined as

$$\delta \phi_{ss'} \triangleq D'_{ss'}(F_{ss'}), \quad \delta \phi_{sj}(w) \triangleq \frac{\partial D_{sj}}{\partial f_{sj}(w)} + \frac{\partial D}{\partial r_j(w)}, \quad \text{and} \quad \delta \phi_{ij}(w) \triangleq \frac{\partial D_{ij}}{\partial f_{ij}(w)} + \frac{\partial D}{\partial r_j(w)}. \quad (17)$$

The partial derivatives in (17), representing the marginal link costs and marginal node costs of a virtual session, are computed as follows.

$$\frac{\partial D_{ij}}{\partial f_{ij}(w)} = D'_{ij}(F_{ij}) \frac{\partial F_{ij}}{\partial f_{ij}(w)}, \quad (18)$$

$$\frac{\partial D}{\partial r_j(w)} = \begin{cases} 0, & \text{if } j = w, \\ \sum_{k \in \mathcal{O}(j)} \phi_{jk}(w) \delta \phi_{jk}(w), & \text{otherwise}. \end{cases} \quad (19)$$

The conditions for optimality stated in the following theorem can be checked by individual nodes using their marginal cost indicators.
Theorem 1: For a feasible set of routing variables to induce the optimal coding subgraphs, the following conditions are necessary. For all \( w \in \mathcal{W} \) and \( i \in \mathcal{N} \setminus \{s,w\} \) such that \( t_i(w) > 0 \),
\[
\delta \phi_{ik}(w) \begin{cases} = \lambda_i(w), & \text{if } \phi_{ik}(w) > 0, \\ \geq \lambda_i(w), & \text{if } \phi_{ik}(w) = 0. \end{cases} \tag{20}
\]
For the source node \( s \), define for every \( w \in \mathcal{W} \), \( \lambda_s(w) = \min_{j \in \mathcal{O}(s)} \delta \phi_{sj}(w) \), then
\[
\delta \phi_{sk}(w) \begin{cases} = \lambda_s(w), & \text{if } \phi_{sk}(w) > 0, \\ \geq \lambda_s(w), & \text{if } \phi_{sk}(w) = 0, \end{cases} \tag{21}
\]
and
\[
\delta \phi_{ss'} \begin{cases} \geq \sum_{w \in \mathcal{W}} \lambda_s(w), & \text{if } \phi_{ss'} = 0, \\ = \sum_{w \in \mathcal{W}} \lambda_s(w), & \text{if } \phi_{ss'} \in (0,1), \\ \leq \sum_{w \in \mathcal{W}} \lambda_s(w), & \text{if } \phi_{ss'} = 1. \end{cases} \tag{22}
\]
The above conditions are sufficient if (20) holds at all intermediate nodes whether \( t_i(w) > 0 \) or not.

To prove the above theorem, we need the following lemma relating the marginal link costs and the marginal source node cost. Its proof is omitted here due to space limitations.

Lemma 1: With link-based and node-based marginal routing costs defined as in (18) and (19), we have for all \( w \in \mathcal{W} \),
\[
\sum_{(i,k) \in \mathcal{E}} \frac{\partial D_{ik}}{\partial f_{ik}(w)} \cdot f_{ik}(w) = \frac{\partial D}{\partial r_s(w)} \cdot R. \tag{23}
\]

Proof of Theorem 1: The necessity of the optimality conditions can be verified in a straightforward manner. We thus prove only the sufficiency part. Assume a set of valid routing configurations \( \{\phi^*_s, \phi^*_i(w)\} \) satisfying the conditions specified in the theorem. Let \( \{\phi^1_s, \phi^1_i(w)\} \) be any other valid routing variables. Denote the resulting flows by \( \{F^*_s, F^*_i(w)\} \) and \( \{F^1_s, F^1_i(w)\} \), respectively. We focus on the difference of the objective values under these two schemes. By the convexity of cost functions \( D_{ij} \) and \( D_{ss'} \) in \( (f_{ij}(w)) \) and \( F_{ss'} \), we have
\[
\sum_{(i,k) \in \mathcal{E}} D_{ik}(F^1_{ik}) + D_{ss'}(F^1_{ss'}) - \sum_{(i,k) \in \mathcal{E}} D_{ik}(F^*_i) - D_{ss'}(F^*_s) > \sum_{(i,k) \in \mathcal{E}} \sum_{w \in \mathcal{W}} \frac{\partial D_{ik}}{\partial f_{ik}(w)} (f_{ik}^1(w) - f_{ik}^*(w)) + \frac{dD_{ss'}}{dF^*_s} (F^1_{ss'} - F^*_s) \]
\[
(a) \sum_{w \in \mathcal{W}} \sum_{(i,k) \in \mathcal{E}} \frac{\partial D_{ik}}{\partial r^*_k} \left[ t^1_k(w) - t^*_k(w) \phi^*_ik(w)\right] \]
\[
(b) R \left( \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{O}(s)} \phi^1_{sk}(w) \delta \phi^*_sk(w) + \phi^1_{ss'} \delta \phi^*_ss' \right) - \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{O}(s)} \phi^*_sk(w) \delta \phi^*_sk(w) + \phi^*_ss' \delta \phi^*_ss' \]
\[
+ \sum_{w \in \mathcal{W}} \sum_{i \neq s,w} t^1_{ik}(w) \left[ \sum_{k \in \mathcal{O}(i)} \phi^*_ik(w) - t^*_ik(w) \phi^*_ik(w)\right]. \tag{24}
\]
Equation (a) is obtained by using Lemma 1 and appending the zero terms $[t^1_i(w) - \sum_i t^1_i(w)\phi^1_{ik}(w)]$. After grouping similar terms and using definitions (17) and (19), we arrive at (b). We next show that $S^1 \geq S^*$ by considering the following two cases.

Case 1: $\phi^*_{ss'} < 1$. This implies that for each $w$, there exists at least one $k \in O(s)$ such that $\phi^*_{sk}(w) > 0$. Then by optimality conditions (21) and (22), $S^* = \sum_{w \in W} \lambda^*_s(w)$, and we have $\delta \phi^*_{ss'} \geq \sum_{w \in W} \lambda^*_s(w)$. Therefore

$$S^1 \geq \sum_{w \in W} \sum_{k \in O(s)} \phi^1_{sk}(w)\lambda^*_s(w) + \phi^1_{ss'} \sum_{w \in W} \lambda^*_s(w) = \sum_{w \in W} \lambda^*_s(w) = S^*.$$ 

Case 2: $\phi^*_{ss'} = 1$. The optimality condition (22) implies $\delta \phi^*_{ss'} \leq \sum_{w \in W} \lambda^*_s(w)$. Therefore,

$$S^1 \geq \sum_{w \in W} \sum_{k \in O(s)} \phi^1_{sk}(w)\lambda^*_s(w) + \phi^1_{ss'} \delta \phi^*_{ss'} = \sum_{w \in W} \lambda^*_s(w)(1 - \phi^1_{ss'}) + \phi^1_{ss'} \delta \phi^*_{ss'} \geq \delta \phi^*_{ss'} = S^*.$$ 

Following similar reasoning, one can verify that the expression in (24) is also non-negative. Thus, we have shown that for any other routing configuration $\{\phi^{*'}_{ss'}, \phi^1_i(w)\}$,

$$\sum_{(i,k) \in E} D_{ik}(F^1_{ik}) + D_{ss'}(F^1_{ss'}) - \sum_{(i,k) \in E} D_{ik}(F^*_{ik}) - D_{ss'}(F^*_{ss'}) > 0.$$ 

Therefore, $\{\phi^{*'}_{ss'}, \phi^1_i(w)\}$ satisfying (20)-(22) must be optimal.

\[\square\]

IV. NODE-BASED DISTRIBUTED ALGORITHMS

After obtaining the optimality conditions, we come to the question of how individual nodes can adjust their local routing variables to find the optimal coding subgraphs. Since the JOCR problem in (9) involves the minimization of a convex objective over convex regions, the class of scaled gradient projection algorithms is appropriate for providing a distributed solution. Using this method, Bertsekas et al. [2] developed distributed routing algorithms for networks supporting unicast sessions. In this section, we adapt this technique to design algorithms for adjusting all virtual sessions’ routing configurations to find the minimum-cost coding subgraphs. These distributed algorithms are used at the source node and intermediate nodes, respectively. Our scheme uses a different technique for computing the scaling matrices and step sizes. This scheme allows us to guarantee the convergence of the algorithms from all initial conditions.

A. Source Node Congestion Control/Routing Algorithm (CR)

The unique source node $s$ controls $\phi_s = (\phi_{ss'}, (\phi_s(w)))$, i.e. it adjusts the admission rate of the multicast session (through $\phi_{ss'}$) and the routing allocations of the incoming traffic with respect to all destinations (through $\phi_s(w)$). Therefore, we specifically call the source node’s algorithm the Congestion control/Routing (CR) algorithm. At the $k$th iteration, the feasible set of vector $\phi_s$ is

$$\mathcal{F}^k_{\phi_s} = \{\phi_s \geq 0 : \phi_{ss'} + \phi_s(w) = 1 \text{ and } \phi_{sj}(w) = 0, \forall j \in B^k_s(w), w \in W\},$$

where 0 denotes the all-zero vector of dimension $|W| \times |O(s)| + 1$, and 1 represents the all-one vector of dimension $|O(s)|$. The notation $B^k_s(w)$ stands for the blocked node set of node $s$ relative to session $w$. This device was invented in [1], [2] to prevent the formation of loops in the routing pattern of session $w$’s traffic. For the source or an intermediate node $i$, $B^k_s(w)$ consists of its neighboring node $j$ with marginal cost $\frac{\partial D}{\partial r^i_j(w)}$ higher than $\frac{\partial D}{\partial r^i_s(w)}$, and neighbors that route positive flows to more costly downstream nodes. By blocking such nodes, we force each session’s traffic to flow through nodes in decreasing order of marginal
costs, thus precluding the existence of loops. For the exact definition of \(B^k(w)\), see [2]. Finally, it is easily seen from its definition that \(F^k_{\phi_s}\) is compact and convex.

Node \(s\) updates the current routing vector \(\phi^k_s\) via the following scaled gradient projection algorithm:

\[
\phi^{k+1}_s = CR(\phi^k_s) = [\phi^k_s - (M^k_s)^{-1} \cdot \delta \phi^k_s]_{M^k_s}^+.
\]  

(25)

Here, \(\delta \phi^k_s\) is the vector of marginal cost indicators \((\delta \phi^k_{ss'}, (\delta \phi^k_s(w)))\), and \(M^k_s\) is a symmetric and positive definite matrix on the subspace

\[
V^k_s = \left\{ v_s : v_{ss'} + \sum_{j \in O(s)} v_{sj}(w) = 0 \text{ and } v_{sj}(w) = 0, \forall w \in W, j \in B^k_s(w) \right\}.
\]

The scaling matrix \(M^k_s\) is specified later. The operator \([\cdot]_{M^k_s}^+\) denotes projection on the feasible set \(F^k_{\phi_s}\) relative to the norm induced by matrix \(M^k_s\). This is given by

\[
[\tilde{\phi}]_{M^k_s}^+ = \arg \min_{\phi \in F^k_{\phi_s}} \langle \phi - \tilde{\phi}, M^k_s(\phi - \tilde{\phi}) \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the standard Euclidean inner product.

### B. Intermediate Node Routing Algorithm (RT)

An intermediate node \(i \neq s, w\) changes the allocation of session \(w\)'s traffic on its outgoing links \(j \in O(i)\) locally by adjusting its current routing vector \(\phi^k_i(w)\) within the feasible set

\[
F^k_{\phi_i(w)} = \left\{ \phi_i(w) \geq 0 : \phi_i(w)' \cdot 1 = 1 \text{ and } \phi_{ij}(w) = 0, \forall j \in B^k_i(w) \right\}.
\]

Here, vectors 0 and 1 are both dimension \(|O(i)|\), and \(B^k_i(w)\) is the blocked node set discussed above. Because \(\phi^k_i(w)\) affects only the routing pattern of session \(w\)'s traffic inside the network, we refer to the updating algorithm at an intermediate node as a pure Routing algorithm (RT). Similar to \(CR\), it has a scaled gradient projection form

\[
\phi^{k+1}_i(w) = RT(\phi^k_i(w)) = [\phi^k_i(w) - (M^k_i(w))^{-1} \cdot \delta \phi^k_i(w)]_{M^k_i(w)}^+.
\]

(26)

Here, \(\delta \phi^k_i(w) = (\delta \phi^k_{ij}(w))\) and \(M^k_i(w)\) is a symmetric and positive definite matrix on the subspace

\[
V^k_i(w) = \left\{ v_i : \sum_{j \in O(i)} v_{ij}(w) = 0 \text{ and } v_{ij}(w) = 0, \forall j \in B^k_i(w) \right\}^3.\]

A specific choice of the scaling matrix \(M^k_i(w)\) is given later. In contrast to \(CR\) at the source node, \(RT\) updates the routing vector \(\phi_i(w)\) of one session at a time.

### C. Marginal Cost Exchange Protocol

In order to let each node acquire the necessary information \(\delta \phi_i(w)\) to implement either \(CR^k\) or \(RT\), protocols for exchanging control messages must be developed. In [1], the rules for propagating the marginal cost information are specified. Before iterating its local algorithm, node \(i\) collects local measures \(\partial f_{ijk}/\partial r_{ijk}(w)\), and inquires its next-hop neighbors \(k \in O(i)\) for their marginal costs \(\partial f_{ik}/\partial r_{ik}(w)\) with respect to the adjusted session(s) \(w\). It then evaluates the terms \(\delta \phi_{ik}(w)\) by using (17).

To update \(\partial f_{ik}(w)\) throughout the network, all nodes compute locally via the recursive equations (19) based on reports from their downstream neighbors, and then provide the results to their upstream neighbors. This procedure must terminate because the network consists of a finite number of nodes. Moreover, the algorithms \(CR\) and \(RT\) guarantee the flow pattern of any session is loop-free.

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3Subspaces \(V^k_i(w)\) and \(V^k_s(w)\) are spanned by feasible incremental routing vectors \(\phi_s - \phi^k_s\) and \(\phi_i(w) - \phi^k_i(w)\) where \(\phi_s, \phi^k_s \in F^k_{\phi_s}\) and \(\phi_i(w), \phi^k_i(w) \in F^k_{\phi_i(w)}\), respectively.

4In the case of \(CR\), \(\delta \phi_{ss'} = D^r_{ss'}(F_s(w))\) is a local measure at \(s\) which is not needed elsewhere. Thus, we ignore \(\delta \phi_{ss'}\) when later discussing the marginal cost exchange protocol.
D. Convergence of Algorithms

The scaled gradient projection method seeks to reduce the objective value with each iteration. Because the update direction at every iteration is opposite to the gradient (scaled by a positive definite matrix) with respect to the adjusted variables, it is a descent direction. However, reduction of the objective cost is guaranteed only when appropriate scaling matrices are used. In this subsection, we specify such matrices for CR and RT, respectively. It turns out that the scaling matrix at each node \( i \) depends on the number of nodes in its downstream node set \( D(N_i) \) relative to session \( w \)'s flow at current iteration \( k \). For convenience, introduce notations \( AN_i^k(w) \equiv \Omega(i) \setminus B_i^k(w) \) and \( AN_s^k \equiv \bigcup_w AN_i^k(w) \).

**Lemma 2:** Assume the initial network cost \( D_0 \) is finite. If at each iteration \( k \) of CR, the scaling matrix

\[
M^k_s = \frac{R}{2} \text{diag} \left\{ A_{ss'}(D_0), \left( |W| \left[ A_{sj}(D_0) + |AN_i^k||\mathcal{D}N_s^k(w)|A(D_0) \right] \right)_{w \in W, j \in AN_i^k} \right\},
\]

where

\[
A_{ij}(D_0) = \max_{F : D_j(F) \leq D_0} D''_{ij}(F) \quad \text{and} \quad A(D_0) = \max_{(m,n) \in E} A_{mn}(D_0),
\]

then the cost is strictly reduced by the current iteration unless the equilibrium conditions (21) and (22) are satisfied at \( s \).

**Proof:** By the Projection Theorem [17], the cost difference after the current iteration is

\[
D(\phi^{k+1}_s) - D(\phi^k_s) = (R\delta \phi^k_s)'(\phi^{k+1}_s - \phi^k_s) + \frac{1}{2} (\phi^{k+1}_s - \phi^k_s)' H^{k,\lambda}_s (\phi^{k+1}_s - \phi^k_s) \leq (\phi^{k+1}_s - \phi^k_s)' - R M^k_s + \frac{H^{k,\lambda}_s}{2} (\phi^{k+1}_s - \phi^k_s),
\]

where \( H^{k,\lambda}_s \) is the Hessian matrix of \( D \) with respect to components of \( \phi_s \), evaluated at \( \lambda \phi^k_s + (1 - \lambda) \phi^{k+1}_s \) for some \( \lambda \in [0,1] \). We temporarily assume that \( D(\phi^{k+1}_s) - D(\phi^k_s) \leq 0 \).

We validate this assumption by showing that \( -R M^k_s + \frac{H^{k,\lambda}_s}{2} \) is negative definite. That is, we show that for all non-zero \( v_s \in V^k_s, v'_s, H^{k,\lambda}_s v'_s \cdot v_s < v'_s \cdot (2RM^k_s) \cdot v_s \). For brevity, we suppress superscripts \((k,\lambda)\) in what follows. Plugging in expressions of all entries of \( H_{\phi_s} \), we have

\[
v'_s \cdot H_{\phi_s} \cdot v_s = \mathbb{R}^2 \left[ D''_{ss'}(F_{ss'}) v_{ss'}^2 + \sum_{w,w'j \in AN_s} \sum_{j \in AN_s} \frac{\partial^2 D_{sj}}{\partial f_{sj}(w) \partial f_{sj}(w')} v_{sj}(w) v_{sj}(w') \right.
\]

\[
\left. + \sum_{w,w'j \in AN_s} \frac{\partial^2 D}{\partial r_j(w) \partial r_j(w')} v_{sj}(w) v_{sk}(w') \right]
\]

\[
\leq (a) \mathbb{R}^2 \left[ D''_{ss'}(F_{ss'}) v_{ss'}^2 + \sum_{j \in AN_s} \left( \sum_{w \in W} \sqrt{\frac{\partial^2 D_{sj}}{\partial f_{sj}(w)^2}} |v_{sj}(w)| \right)^2 \right.
\]

\[
\left. + \left( \sum_{w \in W} \sum_{j \in AN_s} \sqrt{\frac{\partial^2 D}{\partial r_j(w)^2}} |v_{sj}(w)| \right)^2 \right]
\]

\[
\leq (b) \mathbb{R}^2 \left[ D''_{ss'}(F_{ss'}) v_{ss'}^2 + \sum_{j \in AN_s} |W| \sum_{w \in W} \frac{\partial^2 D_{sj}}{\partial f_{sj}(w)^2} v_{sj}^2(w) \right.
\]

\[
\left. + |W||AN_s| \sum_{w \in W} \sum_{j \in AN_s} \sqrt{\frac{\partial^2 D}{\partial r_j(w)^2}} v_{sj}^2(w) \right].
\]

(30)
Inequality (a) follows from $D_{sj}$ being strictly convex in $(f_{sj}(w))$ so that for $w \neq w'$ or $j \neq k$,
\[
\left| \frac{\partial^2 D_{sj}}{\partial f_{sj}(w) \partial f_{sj}(w')} \right| < \frac{\partial^2 D_{sj}}{\partial f_{sj}(w)^2} \frac{\partial^2 D_{sj}}{\partial f_{sj}(w')^2}.
\]
By the Cauchy-Schwarz Inequality, we obtain (b).

It can be shown that
\[
\frac{\partial^2 D}{\partial r_j(w)^2} \leq \max_{(m,n) \in E} \frac{\partial^2 D_{mn}}{\partial f_{mn}(w)^2} |D\mathcal{N}_j(w)|,
\]
where
\[
\frac{\partial^2 D_{mn}}{\partial f_{mn}(w)^2} = D''_{mn}(F_{mn}) \left( \frac{f_{mn}(w)}{F_{mn}} \right)^{2n-2} + D'_{mn}(F_{mn}) \frac{\partial^2 F_{mn}}{\partial f_{mn}(w)^2}.
\]
By relation (10), $f_{mn}(w)/F_{mn} \leq 1$ and
\[
\frac{\partial^2 F_{mn}}{\partial f_{mn}(w)^2} = \frac{n-1}{f_{sj}(w)} \left( \frac{f_{sj}(w)}{F_{sj}} \right)^{n-1} \left[ 1 - \left( \frac{f_{sj}(w)}{F_{sj}} \right)^n \right] \approx 0,
\]
because for $n$ large, either $(f_{sj}(w)/F_{sj})^{n-1} \approx 0$ or $(f_{sj}(w)/F_{sj})^n \approx 1$. Therefore, $\frac{\partial^2 D_{mn}}{\partial f_{mn}(w)^2} \leq D''_{mn}(F_{mn})$. By the assumption $D(\phi_{s}^{k+1}) - D(\phi_{s}^{k}) \leq 0$, we have $D_{mn}(F_{mn}^{k}) \leq D_{0}$ and $D_{mn}(F_{mn}^{k+1}) \leq D_{0}$, so $D_{mn}(\lambda F_{mn}^{k} + (1-\lambda)F_{mn}^{k+1}) \leq D_{0}$ for all $\lambda \in [0,1]$. Accordingly, $D''_{mn}(F_{mn}^{k,k}) \leq A_{mn}(D_{0})$ and $\max_{(m,n) \in E} \frac{\partial^2 D_{mn}}{\partial f_{mn}(w)^2} \leq A(D_{0})$. Substituting the above bounds into (30), we obtain the desired relation $\mathbf{v}_{s} \cdot H_{\phi_{s}^{k},\lambda} \mathbf{v}_{s} < \mathbf{v}_{s} \cdot (2RM_{s}^{k}) \mathbf{v}_{s}$.

Now with the scaling matrix $M_{k}$ specified in the lemma, $D(\phi_{s}^{k+1}) - D(\phi_{s}^{k}) \leq 0$, where the inequality is strict if and only if $\phi_{s}^{k+1} \neq \phi_{s}^{k}$. Moreover, $\phi_{s}^{k} = CR(\phi_{s}^{k})$ only when conditions (21)-(22) hold at node $s$. Thus, the lemma is proved.

Using similar techniques, we can derive appropriate scaling matrices for the algorithms $RT$ used at intermediate nodes.

**Lemma 3:** Assume the initial network cost $D_{0}$ is finite. If at each iteration $k$ of $RT$, the scaling matrix
\[
M_{k}(w) = \frac{t_{k}(w)}{2} \text{diag} \left\{ A_{ij}(D_{0}) + |\mathcal{AN}_{k}^{i}(w)||D\mathcal{N}_{j}^{k}(w)|A(D_{0}) \right\}_{j \in \mathcal{AN}_{k}^{i}(w)},
\]
then the cost is strictly reduced by the current iteration unless the equilibrium condition (20) is satisfied by $\phi_{s}^{k}(w)$.

Building on the above two lemmas, we have the following convergence theorem.

**Theorem 2:** Assume an initial congestion control ratio $\phi_{0}^{ss'}$ and loop-free routing configuration $\{\phi_{i}^{i}(w)\}$ such that $D(\phi_{ss'}, \{\phi_{i}^{i}(w)\}) = D_{0} < \infty$. If the scaling matrices are chosen according to Lemmas 2 and 3, then the sequences generated by algorithms $CR$ and $RT$ converge, i.e. $\phi_{ss'}^{k} \rightarrow \phi_{ss'}^{*}$ and $\phi_{i}^{i}(w) \rightarrow \phi_{i}^{*}(w)$ for all $w \in \mathcal{W}$ and $i \neq w$ as $k \rightarrow \infty$. Furthermore, $\phi_{ss'}^{*}$ and $\{\phi_{i}^{*}(w)\}$ constitute a set of jointly optimal solution of JOCR (9).

**Proof:** With the scaling matrices specified in Lemmas 2 and 3, any iteration of $CR$ and $RT$ strictly reduces the total network cost with all other variables fixed, unless the equilibrium conditions for the adjusted variables are satisfied. Because optimization variables $\phi_{s}$ and $\{\phi_{i}^{i}(w)\}$ each takes values in a compact set, the sequences $\{\phi_{s}^{k}\}_{k=0}^{\infty}$ and $\{\phi_{i}^{i}(w)\}_{k=0}^{\infty}$ must each have a convergent subsequence. As the objective function is bounded below, the non-increasing sequence of network cost generated by iterations of all the algorithms must converge. Therefore, the limit points $\phi_{s}^{*}$ and $\phi_{i}^{*}(w)$ of above sequences must be such that
conditions (21)-(22) are satisfied by $\phi^*$ and condition (20) is satisfied by $\phi_i(w)$ at all $i \neq s, w$. Therefore by Theorem 1, these limit points jointly constitute an optimal solution of JOCR (9).

Note that global convergence does not require any particular order in running the algorithms $CR$ and $RT$ at different nodes. For convergence to the joint optimum, every node $i$ only needs to iterate its own algorithm(s) until its routing variables satisfy either (21)-(22) or (20). In practice, nodes may keep updating their routing variables with the corresponding algorithms until further reduction in network cost by any one of the algorithms is negligible.

V. Conclusion

We adopt the network coding approach to achieve minimum-cost multicast. In light of the intrinsic similarity between the optimal coding subgraph problem and the conventional multi-commodity flow routing problem, we apply distributed routing algorithms to solve the multicast subgraph optimization. We develop a node-based optimization framework, and derive the necessary and sufficient optimality conditions for convex link costs and concave utility functions. Finally, we design a complete set of distributed algorithms involving both congestion control and routing, and prove its convergence to the optimum network configuration.

REFERENCES